How to calculate the probability mass function of a train of random pulses each containing a random number of counts. (this could be generalized to some random continuous response.)

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This follows certain standard mathematical procedure from the theory of probability.

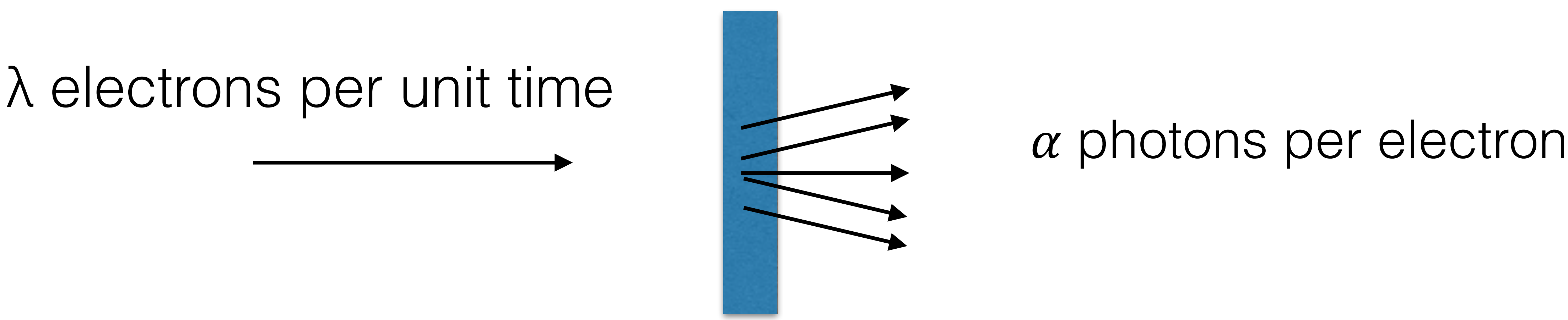
Question

Mean of λ electrons come to the electroluminescence region

Each electron generates α photons.

What is the mean number of photons ?

What is the variance on the number of photons ?



Other questions are similar. If mean of λ photons convert in a photo-sensor with a mean gain of α electrons per photon what is the distribution of the output number of electrons.

Basically, an average of λ packets arrive each with an average of α items in each packet. What is the mean and variance of the total number of items ?

Distribution of incoming electrons with Poisson mean of λ

K is the number of electrons, a discrete random variable with probability mass function

$$\mathrm{P}_K(\mathrm{k}) = \frac{\lambda^k}{k!}e^{-\lambda}$$

Generating function for this is (think of s^k as a tag for the value of the probability)

$$G_K(s) \equiv \sum_{k=0}^{\infty} s^k P_K(k) = \sum_{k=0}^{\infty} \frac{s^k \lambda^k}{k!} e^{-\lambda} = e^{s\lambda} e^{-\lambda} = e^{\lambda(s-1)}$$

G_K is also expressed as the expectation value for s^k

$$\mathrm{G}_K(s) = E[s^k]$$

Probability is a sequence of numbers and $\mathrm{G}_K(s)$ allows us to organize this sequence in a compact manner. The series always converges for $|s| \leq 1$.

Similarly, the distribution of photons from each electron has probability

$\mathrm{P}_L(\ell) = \dfrac{\alpha^\ell}{\ell!}e^{-\alpha}$ and a similar generating function $\mathrm{G}_L(s)$.

We have labeled the two generating functions to distinguish them from each other.

Additionally recall that the mean for a Poisson distribution with parameter λ is $E[k] = \lambda$ and the variance is also $E[k^2] - (E[k])^2 = \lambda$

Probability for total photon count

Total number of photons is a discrete random number Z.

$$Z = \sum_{i=1}^{K} L_i$$

This is a sum of K random numbers, each is the photon count from an electron.
Now, it is obvious that the mean number of total photons must be $\lambda \times \alpha$, where λ is the Poisson mean for the number of electrons and α is the Poisson mean for the number of photons emitted by each electron.
However, the random number for the total number of photons is not a product of the two random numbers K (the number of electrons) and L (the number of photons).
What we are trying to calculate is the generating function for the total number of photons and consequently the probability mass function for the total number of photons.

$$G_Z(s) \equiv E[s^z] = \sum_{z=0}^{\infty} s^z P_Z(Z = z)$$

Here P_Z is unknown. We need to calculate the mean and variance of P_Z.

Expectation values

Start with the generating function for total number of photons

(recall that K is the r.v. for electrons and L is the r.v. for photons/electron)

$$G_Z(s) \equiv \sum_{z=0}^{\infty} s^z P_Z(Z=z) = E[s^z] = \sum_{k=0}^{\infty} E[s^{\sum_{i=1}^{K} L_i} \,|\, K=k] \bullet P_K(k)$$

$$= \sum_{k=0}^{\infty} E[s^{L_1} s^{L_2} \ldots s^{L_K} \,|\, K=k] \bullet P_K(k)$$

This says that the total expectation for s^z is the same as

the average of the conditional expectation for k electrons (averaged over the

probability of obtaining k electrons). This is the law of total expectation.

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Imagine that outcomes {a,b,c} have probability {0.1, 0.2, 0.7}, respectively.

For each outcome we get an average of {5, 7, 10} dollars.

Then the expectation for the total amount of money is

$5 \times 0.1 + 7 \times 0.2 + 10 \times 0.7 = 8.9$

The dollar value for each outcome could also have a distribution, but

we are averaging over it to get the total expectation.

Conditional expectation

And so we have to first calculate the conditional expectation

$$E[s^{L_1}s^{L_2}\dots s^{L_K}\,|\,K=k]=\sum_{\ell_1,\ell_2\dots\ell_K=0}^{\infty} s^{\ell_1}s^{\ell_2}\dots s^{\ell_k}P_L(\ell_1)P_L(\ell_2)\dots P_L(\ell_k)$$

since all P_L are identical Poisson, the sums are identical, and therefore

$$E[s^{L_1}s^{L_2}\dots s^{L_K}\,|\,K=k]=\left\{\sum_{\ell=0}^{\infty} s^{\ell}P_L(\ell)\right\}^k=(G_L(s))^k$$

$$E[s^{L_1}s^{L_2}\dots s^{L_K}\,|\,K=k]=(e^{\alpha(s-1)})^k$$

To be clear, this is the generating function for the number of photons under the condition that exactly k electrons emitted photons in the electroluminescent region.

Or that we have exactly k packets each with a random number of items $\{\ell_1,\ell_2,\dots,\ell_k\}$.
The sum of this array of packets has the probability encoded in the above generating function

Now we have to calculate the average of this expectation over the number of incoming electrons.

Total expectation

Now we calculate the expectation of this conditional expectation.

$$E[s^z] = \sum_{k=0}^{\infty} (e^{\alpha(s-1)})^k \times P_K(k) = \sum_{k=0}^{\infty} (f)^k \times P_K(k) = e^{\lambda(f-1)} = e^{\lambda(e^{\alpha(s-1)}-1)}$$

Notice that this could be succinctly restated as

$G_Z(s) = G_K(G_L(s))$ .... if this is confusing notice that the argument for G_K is G_L

And so we have the generating function

$$G_Z(s) = e^{\lambda(e^{\alpha(s-1)}-1)}$$

We can now use this to obtain the mean, variance, and the probability mass function for the total number of photons. Notice that the function is not symmetric in λ and α.

One would incorrectly think that it should be if we naively think of the total number of photons as a random variable that is a product of two independent random variables.

It is easy to obtain the mean and variance using these formulas.

$$E[z] = \left.\frac{dG_Z(s)}{ds}\right|_{s=1} = \lambda e^{\lambda(e^{\alpha(s-1)}-1)} \alpha\, e^{\alpha(s-1)}\Big|_{s=1} = \lambda\alpha \qquad \text{as expected.}$$

$$Var[z] = E[z^2] - (E[z])^2 = \left(\frac{d^2 G_Z(s=1)}{ds^2} + E[z]\right) - (E[z])^2 \quad \text{... gets a little tricky here.}$$

Each derivative of G_Z pulls down z, $(z-1)$, etc. Recall that $G_Z(s) = E[s^z]$.

$$Var[z] = \lambda\alpha \cdot e^{\alpha(s-1)+\lambda(e^{\alpha(s-1)}-1)} \cdot (\alpha + \lambda\alpha \cdot e^{\alpha(s-1)})\Big|_{s=1} + \lambda\alpha - \lambda^2\alpha^2$$

$$Var[z] = \lambda\alpha(\alpha+1)$$

Probability mass function

Notice that the variance is in fact not symmetric in λ and α and it is always larger than the variance expected from a Poisson probability mass function with a mean of $\alpha\lambda$.

$Var[z] = \lambda\alpha(\alpha+1)$

It is best to understand this using the full probability mass function which we will now calculate. We have to invert $G_Z(s) \equiv \sum_{z\geq 0} s^z P_Z(z)$ to obtain P_Z where $z \geq 0$ is an integer.

It is easiest to do this using Fourier series with the substitution $s \rightarrow e^{is}$

$\varphi_Z(s) = G_Z(e^{is}) = e^{\lambda(e^{\alpha(e^{is}-1)}-1)}$

This is called the characteristic function. There is a formal mathematical relationship here which we will not explain. Most importantly, the characteristic function of a probability mass function always exists and is finite. Now the inverse is easily written down as this

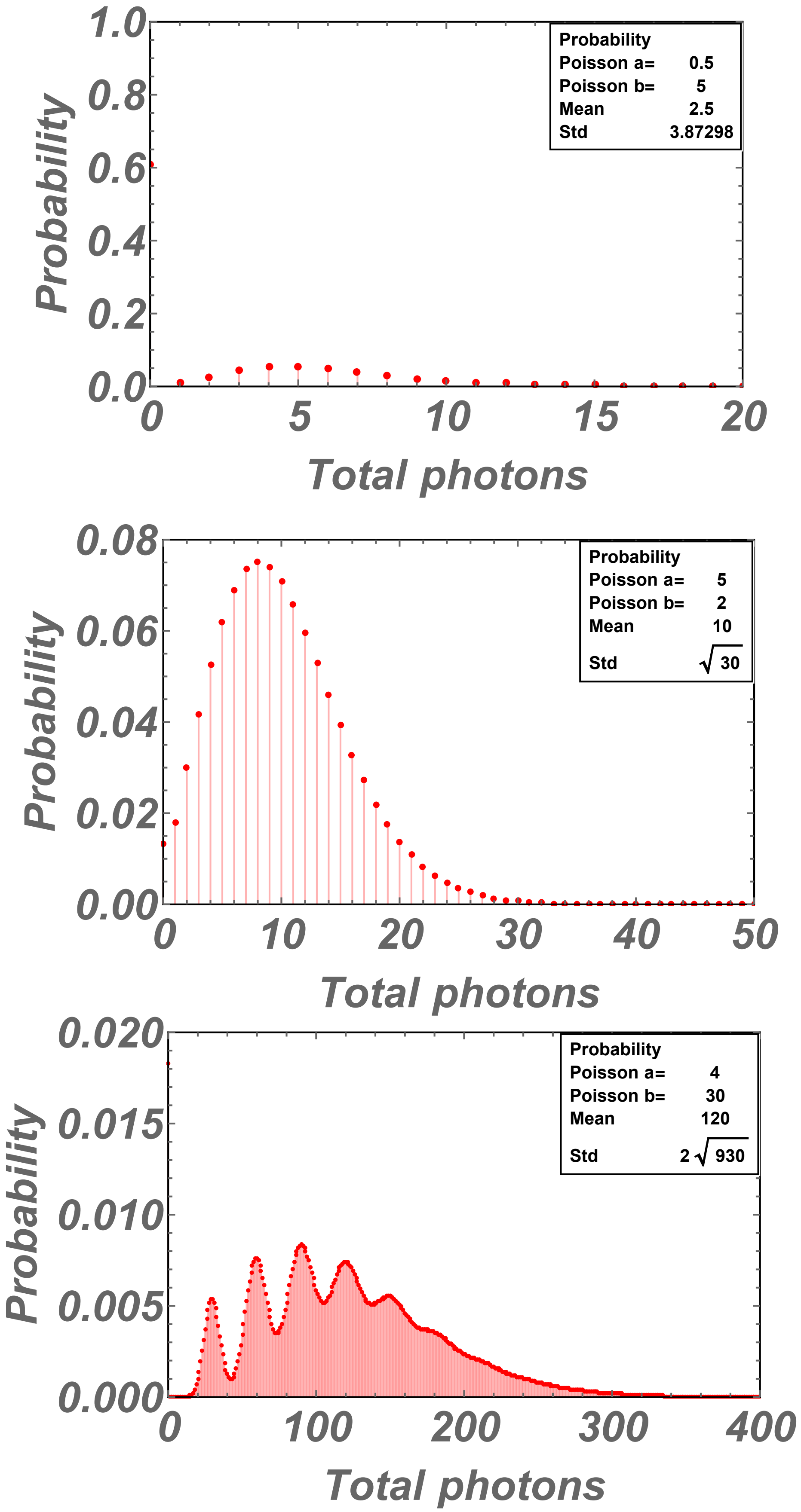
$$P_Z(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi_Z(s)\cdot e^{-i\cdot s\cdot z}\, ds$$

Notice that z is an integer and φ_Z is an infinite series with each probability tagged by a positive integer power of e^{is}.

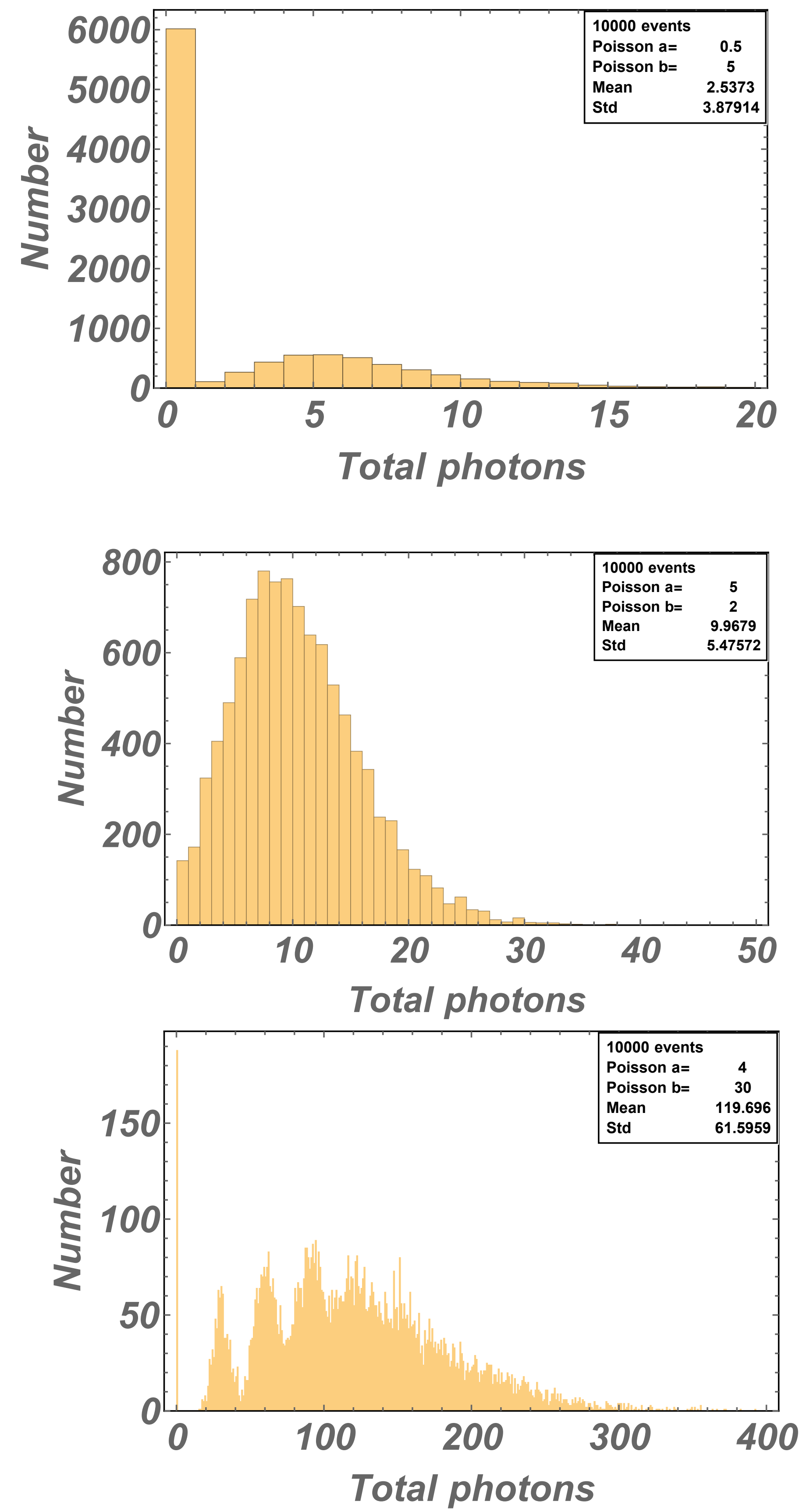
Over the inteval $[-\pi,\pi]$, any term that does not have the same power as z will get zeroed out. This is just like a Fourier series. The integral can be calculated numerically for specific values of λ and α and z. The full form is here.

$$\boxed{P_Z(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\lambda(e^{\alpha(e^{is}-1)}-1)}\cdot e^{-i\cdot s\cdot z}\, ds \quad \text{where } z \geq 0 \text{ is an integer}}$$

Examples.



Calculation, mean and std from formula



Monte Carlo, mean/standard deviation from simulated data

The above could be generalized to some arbitrary discrete or continuous response function. This response function is called the "jump" (J) distribution.
The resulting distribution is called a sputtering-Poisson (SP) distribution (and other names)
And so if a mean number of λ particles convert in a detector with some response distribution such as the Gaussian, Exponential, or Landau distribution how can we calculate the mean and variance ?

	Jump PDF	Charac. func. $\Phi_J(t)$
Gaussian	$\frac{1}{\sqrt{2\pi\sigma}} e^{-(x-\mu)/2\sigma^2}$	$e^{it\mu} \bullet e^{-\sigma^2 t^2/2}$
Exponential	$\frac{1}{\alpha} e^{-\alpha x}$	$\frac{1}{(1-\frac{it}{\alpha})}$
Landau	$L(\mu,c)$	$e^{it\mu} \bullet e^{-\lvert ct\rvert(1+\frac{2i}{\pi}Log\lvert t\rvert)}$

For a sputtering Poisson with Gaussian as jump. We will call the random number X_{SG}

$$\varphi_{SG}(t) = e^{\lambda\left[e^{it\mu}\cdot e^{-\sigma^2 t^2/2}-1\right]} \quad \ldots\ldots$$ The PDF can be obtained by an inverse Fourier transform.

$$\langle x_{SG}\rangle = -i\frac{d\varphi_{SG}}{dt}(t=0) = \mu\cdot\lambda \quad \ldots\ldots \text{ as expected.}$$

$$\langle x_{SG}^2\rangle = (-i)^2\frac{d^2\varphi_{SG}}{dt^2}(t=0) = \lambda(\sigma^2+(\lambda+1)\mu^2)$$

$$Var(x) = \langle x_{SG}^2\rangle - \langle x_{SG}\rangle^2 = \lambda(\sigma^2+\mu^2) \quad \ldots\ldots$$ you are welcome to do the algebra.

Assume that there are λ=5 Hz of muons in a detector. And each muon produces a charge pulse that is Gaussian distributed with $(\mu\pm\sigma)(2.00\pm0.02)\,fc$ each.

Then the mean value of the charge each second is $\lambda\mu$=10 fc

The variance on the charge will be $\lambda(\sigma^2+\mu^2)\approx\lambda\mu^2 = 500\,fc^2$

Notice that for small σ, the ratio

$$\frac{Var(x)}{\langle x_{SG}\rangle} = \frac{\lambda(\sigma^2+\mu^2)}{\mu\cdot\lambda} \approx \mu \quad \ldots\ldots$$ Obviously this is only if $\lambda > 0$ and $\mu \gg \sigma$

Thus only from the mean and the variance of a current one could determine the mean response of a detector.

For a sputtering Poisson with Exponential as jump. We will call the random number X_{SE}

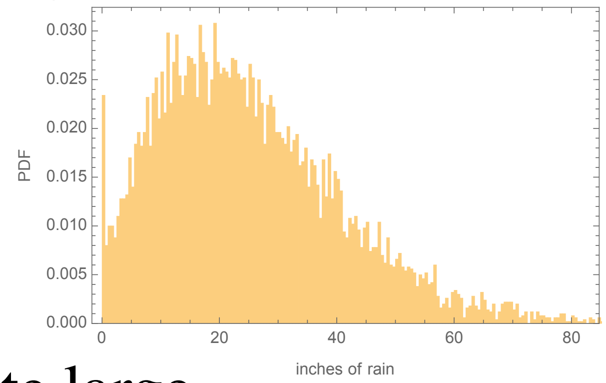
$$\varphi_{SE}(t) = e^{\lambda\left[\frac{it/\alpha}{1-it/\alpha}\right]}$$ ..... The PDF can be obtained by an inverse Fourier transform.

$$\langle x_{SE} \rangle = -i\frac{d\varphi_{SE}}{dt}(t=0) = \frac{\lambda}{\alpha}$$ ..... as expected.

$$\langle x_{SE}^2 \rangle = (-i)^2 \frac{d^2\varphi_{SE}}{dt^2}(t=0) = \frac{\lambda}{\alpha^2}(2+\lambda)$$

$$Var(x_{SE}) = \langle x_{SE}^2 \rangle - \langle x_{SE} \rangle^2 = \frac{2\lambda}{\alpha^2}$$ .... Don't think I have screwed up the algebra

Again there are λ=5 per day of rain showers. And each showers produces a $\alpha^{-1} = 5$ inches of rain with an exponential distribution.



Then the mean value of rain per day is λ / α=25 inches

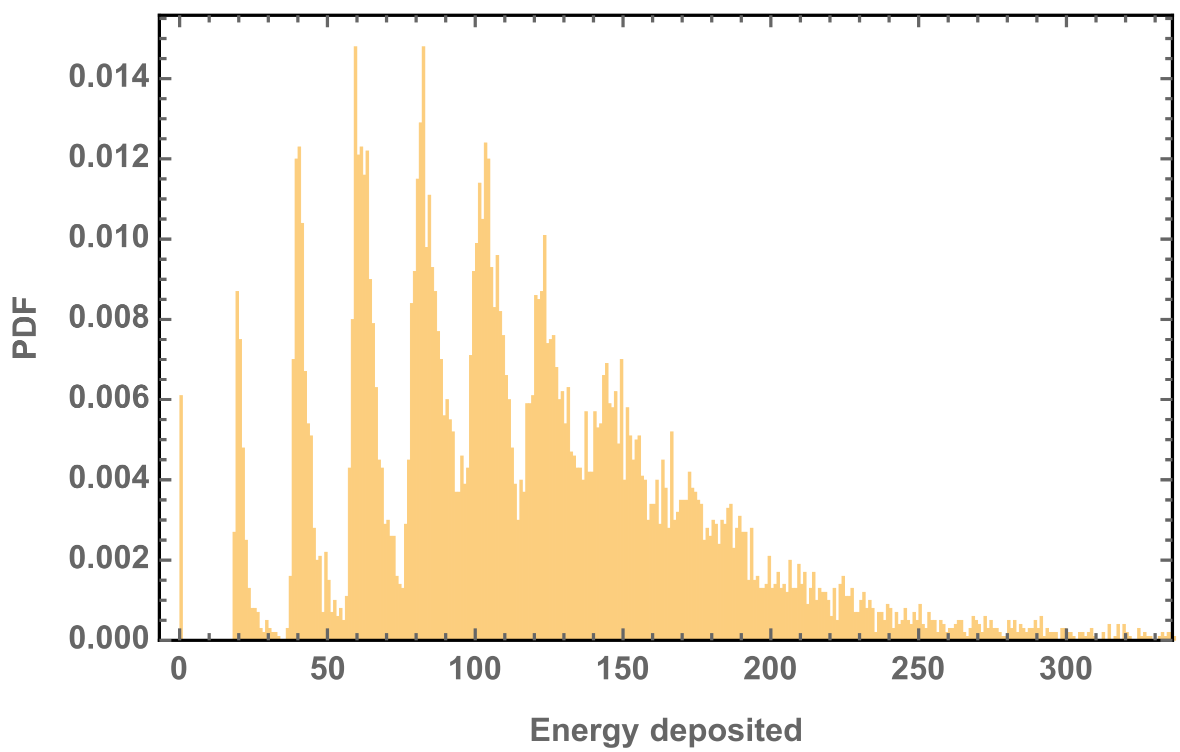
The variance on the rain will be $2\lambda / \alpha^2 = 250\ inch^2 \approx (16\text{inch})^2$ ...quite large

The final distribution is not so simple. It can have many zeros, and has a rise and fall.

For a sputtering Poisson with Landau as jump. We will call the random number X_{SL}

$$\varphi_{SL}(t) = e^{\lambda\left[e^{it\mu}e^{-|ct|(1+\frac{2i}{\pi}Log|t|)}-1\right]}$$ ..... It is possible to simulate this.

The mean and variance are undefined. It is one of those crazy distributions.



Take λ =5

And Landau location parameter μ = 20

and Landau shape parameter c = 1

We will work on various quantities regarding this distribution in subsequent work

$$\text{Main distribution } P_K(k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

Jump PDF	parameters	Mean	Variance
Poisson	$\mathrm{P}_L(\mathrm{k}) = \frac{\alpha^k}{k!} e^{-\alpha}$	$\lambda\alpha$	$\lambda\alpha(\alpha+1)$
Normal	$\frac{1}{\sqrt{2\pi\sigma}} e^{-(x-\mu)/2\sigma^2}$	$\lambda\mu$	$\lambda(\sigma^2+\mu^2)$
Exponetial	$\frac{1}{\alpha} e^{-\alpha x}$	λ/α	$2\lambda/\alpha^2$
Landau	$L(\mu,c)$	undef.	undef